A SHORT PROOF OF GAMAS'S THEOREM

ANDREW BERGET

ABSTRACT. If χ^λ is the irreducible character of $\mathfrak{S}_n$ corresponding to the partition λ of n then we may symmetrize a tensor $v_1 \otimes \cdots \otimes v_n$ by χ^λ. Gamas's theorem states that the result is not zero if and only if we can partition the set $\{v_i\}$ into linearly independent sets whose sizes are the parts of the transpose of λ. We give a short and self-contained proof of this fact.

1. INTRODUCTION

Let λ be a partition of a positive integer n and let χ^λ be the irreducible character of the symmetric group $\mathfrak{S}_n$ corresponding to λ. There is a right action of $\mathfrak{S}_n$ on $V^{\otimes n}$, where V is a finite-dimensional complex vector space, by permuting tensor positions. Let T_λ be the endomorphism of $V^{\otimes n}$ given by

$$(v_1 \otimes \cdots \otimes v_n)T_\lambda = \frac{\chi^\lambda(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi_\lambda(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

Our goal is to prove the following result of Carlos Gamas [3].

Theorem 1 (Gamas's Theorem). Let $v_1, \ldots, v_n$ be vectors in V. Then

$$(v_1 \otimes \cdots \otimes v_n)T_\lambda \neq 0$$

if and only if it is possible to partition the set $\{v_i\}$ into linearly independent sets whose sizes are the parts of the transpose of λ.

If $\{v_1, \ldots, v_n\}$ is a collection of vectors satisfying the condition of the theorem we will say that it satisfies "Gamas's Condition for λ". The theorem is a generalization of the well known fact that the exterior product of a set of vectors is nonzero if and only the set of vectors is linearly independent.

In addition to Gamas's proof of this result there was a second one given by Pate in 1990 [4] using results he obtained in [5]. The benefit of our proof is that it is self-contained and short. It relies on standard facts from the representation theory of $\mathrm{GL}(V)$, namely, Schur-Weyl duality and the Pieri Rule. We refer to Fulton and Harris's book [2] for the needed background and notation.

2. PRELIMINARIES AND PROOF

Let V be a finite dimensional complex vector space. The general linear group $\mathrm{GL}(V)$ acts diagonally on $V^{\otimes n}$. Let $w \in V^{\otimes n}$ be any tensor. Define $G(w)$ to be the $\mathrm{GL}(V)$-module spanned by

$$\mathrm{GL}(V)w = \{g \cdot w : g \in \mathrm{GL}(V)\}.$$

We are interested in which irreducible $\mathrm{GL}(V)$-modules appear in $G(w)$. Since $G(w) \subset V^{\otimes n}$ is a polynomial representation, the isomorphism type of the irreducible $\mathrm{GL}(V)$-modules which can appear in $G(w)$ are indexed by partitions λ of n with at most $\dim V$ parts. If λ is such a partition, we will say that λ *appears in* $G(w)$ if this module has a highest weight vector of weight λ (see [2, Chapter 15]). We will write $\ell(\lambda)$ for the number of parts of λ.

Proposition 2. If λ is a partition of n, then λ appears in $G(w)$ if and only if $wT_\lambda \neq 0$.

Proof. Note that λ appears in $G(w)$ if and only if the projection of $G(w)$ onto its λ-th isotypic component is not zero. By Schur-Weyl duality (see [2, Lemma 6.22]) T_λ is this projector, since it is the projector of $\mathbb{C}\mathfrak{S}_n$ onto its λ-th isotypic component. Since T_λ commutes with the $\mathrm{GL}(V)$ action, this isotypic component is zero if and only if $G(wT_\lambda) = 0$, which happens if and only if $wT_\lambda = 0$. $\qquad\square$

The following corollary is immediate from Proposition 2.

Corollary 3. Suppose that W is a subspace of V and $w \in W^{\otimes n} \subset V^{\otimes n}$. The shape λ appears in $\mathrm{span}\,\mathrm{GL}(V)w$ if and only if it appears in $\mathrm{span}\,\mathrm{GL}(W)w$.

Proof of Gamas's Theorem. Assume that the vectors $\{v_1, \ldots, v_n\}$ span V, as we may by Corollary 3. Suppose that $\{v_i\}$ satisfy Gamas's condition for λ. We prove the result by induction on $n + \ell(\lambda)$. Write $v^\otimes$ for the tensor $v_1 \otimes \cdots \otimes v_n$.

If λ has one part χ^λ is the trivial character and $v^\otimes T_\lambda$ is a scalar multiple of the fully symmetrized tensor $v_1 \cdots v_n$ in $\mathrm{Sym}^n(V)$. This is not zero since none of the v_i are zero.

If $\ell(\lambda) < \dim V$ let $A \in \mathrm{End}(V)$ be a generic projection to a subspace $W \subset V$ with dimension equal to the length of λ. Since A is generic, the collection $\{Av_1, \ldots, Av_n\}$ still satisfies Gamas's condition for λ. It follows by induction that λ appears in the span of $\mathrm{GL}(W)(A \cdot v^\otimes)$ and hence it also appears in $G(A \cdot v^\otimes)$. Since A is a limit of elements of $\mathrm{GL}(V)$ we have $G(A \cdot v^\otimes) \subset G(v^\otimes)$ and hence λ appears in $G(v^\otimes)$.

If $\ell(\lambda) = \dim V$ then we consider a Young tableau of shape λ whose columns index independent subsets of the set $v = \{v_1, \ldots, v_n\}$. Let B be the set of numbers in the first column of the tableau. The map

$$b_B = \sum_{\sigma \in \mathfrak{S}_B} (-1)^\sigma \sigma \in \mathbb{C}\mathfrak{S}_n = \mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes n})$$

is a map of $\mathrm{GL}(V)$-modules and hence we have a surjection of $\mathrm{GL}(V)$-modules $G(v^\otimes) \to G(v^\otimes b_B)$. Without loss of generality, we write $B = \{1, \ldots, k\}$ where $k = \dim V$, so that

$$G(v^\otimes b_B) = \det{}_V \otimes G(v_{k+1} \otimes \cdots \otimes v_n).$$

Here $\det_V$ is the one dimensional representation $g \mapsto \det(g)$ of $\mathrm{GL}(V)$. For example, if $\dim V = 2$ and $B = \{1, 2\}$ then

$$(v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5)b_B = (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes v_3 \otimes v_4 \otimes v_5$$

Since v_1 and v_2 are a basis of V, we see that $g \in \mathrm{GL}(V)$ acts by its determinant on the exterior power $\bigwedge^2 V$ and hence on $v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_1$.

Denote by λ^- the shape obtained from λ by removing the first column. Then $\{v_{k+1}, \ldots, v_n\}$ satisfies Gamas's condition for λ^-. By induction we know that λ^- appears in $G(v_{k+1} \otimes \cdots \otimes v_n)$. By Pieri's Rule (see [2, Equation 6.9]) it follows that λ appears in

$$\det{}_V \otimes G(v_{k+1} \otimes \cdots \otimes v_n)$$

and, hence, in $G(v^\otimes)$ since it appears in its homomorphic image $G(v^\otimes b_B)$. This completes the more difficult implication of Gamas's Theorem.

Although our proof of the converse was essentially known to Pate [4], we include it to keep this paper self-contained. We will need the standard construction of the irreducible $\mathrm{GL}(V)$ and $\mathbb{C}\mathfrak{S}_n$ modules via Young symmetrizers. To this end let T be a tableau of shape λ, a_T its row symmetrizer, and b_T its column antisymmetrizer. These are given by

$$\sum_{\sigma \in \mathrm{Row}(T)} \sigma, \quad \sum_{\sigma \in \mathrm{Col}(T)} \mathrm{sign}(\sigma)\sigma,$$

respectively. For example, using cycle notation for permutations in $\mathfrak{S}_n$, if

$$T = \begin{array}{|c|c|c|} \hline 2 & 3 & 4 \\ \hline 1 & 5 \\ \cline{1-2} \end{array}$$

then $b_T = (1 - (12))(1 - (35))$ while

$$a_T = (1 + (23) + (24) + (34) + (234) + (243))\,(1 + (15)).$$

A product $b_T a_T$ is called a *Young symmetrizer* and the right ideal in $\mathbb{C}\mathfrak{S}_n$ generated by a Young symmetrizer is an irreducible $\mathbb{C}\mathfrak{S}_n$-module with character χ^λ while the image of $b_T a_T$ on $V^{\otimes n}$ is zero, or irreducible with highest weight λ (see [2, Chapters 4 and 15]). It is clear that $v^{\otimes} b_T$ is not zero if and only if the sets of vectors indexed by the columns of the tableau T are linearly independent.

It follows from Schur-Weyl duality that if λ appears in $G(v^{\otimes})$ then there is an element $c \in \mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes n}) = \mathbb{C}\mathfrak{S}_n$ such that $G(v^{\otimes}c)$ equals the irreducible $\mathrm{GL}(V)$-module $V^{\otimes n} b_T a_T$. It then follows that the right $\mathbb{C}\mathfrak{S}_n$-module generated by $v^{\otimes}c$ is isomorphic to both $c\mathbb{C}\mathfrak{S}_n$ and $b_T a_T \mathbb{C}\mathfrak{S}_n$, in particular the latter two modules are isomorphic. We conclude that c can be written as a sum $c = \sum_{\sigma \in \mathfrak{S}_n} x_\sigma \sigma b_T a_T$, $x_\sigma \in \mathbb{C}$, and hence one of these terms $x_\sigma \sigma b_T a_T$ applied to $v^{\otimes}$ is not zero. Finally, since $v^{\otimes}\sigma b_T$ is not zero Gamas's Condition holds for λ, the shape of T. $\qquad\square$

Define a sequence of integers ρ_i by the condition that

$$\rho_1 + \cdots + \rho_k$$

is the size of the largest union of k linearly independent subsets of $\{v_i\}$. The sequence ρ is called the *rank partition* of $\{v_i\}$ and was introduced by Dias da Silva in [1]. In our language, Dias da Silva proved the following strengthening of Gamas's Theorem.

Theorem 4 (Dias da Silva)**.** The partition λ appears in $G(v^{\otimes})$ if and only if λ is larger (in dominance order) than the transposed rank partition of $\{v_i\}$.

The extent to which one can further predict the irreducible $\mathrm{GL}(V)$-decomposition of $G(v^{\otimes})$ is the subject of the author's Ph.D. thesis.

References

[1] J.A. Dias da Silva. On the μ-colorings of a matroid. *Linear and Multilinear Algebra*, 27(1):25–32, 1990.
[2] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.
[3] Carlos Gamas. Conditions for a symmetrized decomposable tensor to be zero. *Linear Algebra Appl.*, 108:83–119, 1988.
[4] Thomas H. Pate. Immanants and decomposable tensors that symmetrize to zero. *Linear and Multilinear Algebra*, 28(3):175–184, 1990.
[5] Thomas H. Pate. Partitions, irreducible characters, and inequalities for generalized matrix functions. *Trans. Amer. Math. Soc.*, 325(2):875–894, 1991.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: berget@math.umn.edu